INVARIANT DISTRIBUTIONS FOR HOMOGENEOUS FLOWS

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Abstract. We prove that every homogeneous flow on a finite-volume homogeneous manifold has countably many independent invariant distributions. As a part of the proof, we have that any smooth partially hyperbolic flow on any compact manifold has countably many distinct minimal sets, hence countably many distinct ergodic probability measures. As a consequence, the Katok and Greenfield-Wallach conjectures hold in all of the above cases.

1. Introduction

A smooth flow (ϕ_t) generated by a smooth vector field X on a compact manifold M is called *stable* if the range of the Lie derivative $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ is closed and it is called *cohomology-free* or *rigid* if it is stable and the range of the Lie derivative operator has codimension one. The *Katok (or Katok-Hurder) conjecture* [Kat01], [Kat03], [Hur85] states that every cohomology-free vector field is smoothly conjugate to a linear flow on a torus with Diophantine frequencies. It is not hard to prove that all cohomology-free flows are volume preserving and uniquely ergodic (see for instance [For08]).

We also recall that the Katok conjecture is equivalent to the *Greenfield-Wallach conjecture* [GW73] that every globally hypoelliptic vector field is smoothly conjugate to a Diophantine linear flow (see [For08]). A smooth vector field X is called *globally hypoelliptic* if any 0-dimensional current U on M is smooth under the condition that the current $\mathcal{L}_X U$ is smooth. Greenfield and Wallach in [GW73] proved this conjecture for homogeneous flows on *compact* Lie groups. (The equivalence of the Katok and Greenfield-Wallach conjectures was essentially proved already in [CC00] as noted by the third author of this paper. The details of the proof can be found in [For08]).

The best general result to date in the direction of a proof is the joint paper of the third author [RHRH06] where it is proved that a every cohomology-free vector field has a factor smoothly conjugate to Diophantine linear flow on a torus of the dimension equal to the first Betti number of the manifold M. This result has been developed independently by several authors [For08], [Koc09], [Mat09] to give a complete proof of the conjecture in dimension 3 and by the first author in the joint paper [FP11] to prove that every cohomology-free flow can be embedded continuously as a linear flow in a possibly non-separated Abelian group.

From the definition, it is clear that there are two main mechanisms which may prevent a smooth flow from being cohomology-free: it can happen that the flow is not stable or it can happen that the closure of its range has codimension higher than one. For instance, linear flows on tori with *Liouvillean* frequencies are not stable with range dense in a closed subspace of dimension one (the subspace of functions of zero average), while translation flows [For97], [MMY05], horocycle flows [FF03] and nilflows [FF07] are in general stable but have range of countable codimension.

Until recently, basically no other examples were known of non cohomology-free smooth flows. In particular there was no example of the first kind of phenomenon, that is, of flows which are not stable with range closure of codimension one, except

for flows smoothly conjugate to *linear* Liouvillean toral examples. In the past couple of years several examples of this kind have been constructed by Avila and collaborators. Avila and Kocsard [AK11] have proved that every non-singular smooth flow on the 2-torus with irrational rotation number has range closed in the subspace of functions of zero average with respect to the unique invariant probability measure. Avila and Kocsard and Avila and Fayad have also announced similar examples on certain higher dimensional compact manifolds, not diffeomorphic to tori, which admit a non-singular smooth circle action (hence Conjecture 6.1 of [For08] does not hold). In all these examples the closure of the range of the Lie derivative operator on the space smooth functions has codimension one, or equivalently, the space of all invariant distributions for the flow has dimension one (hence it is generated by the unique invariant probability measure). We recall that an *invariant distribution* for a flow is a distribution (in the sense of S. Sobolev and L. Schwartz) such that its Lie derivative along the flow vanishes in the sense of distributions.

The goal of this paper is to prove that examples of this kind do not exist among *homogeneous flows*, so that a non-toral homogeneous flow always fails to be cohomology-free also because the closure of its range has codimension higher then one. In fact, we prove that for any homogeneous flow on a *finite-volume* homogeneous manifold M, except for the case of flows smoothly isomorphic to linear toral flows, the closure of the range of the Lie derivative operator on the space of smooth functions has countable codimension, or, in other terms, the space of invariant distributions for the flow has countable dimension. As a corollary we have a proof of the Katok and Greenfield-Wallach conjectures for general homogeneous flows on finite-volume homogeneous manifolds. Our main result can be stated as follows.

Theorem 1.1. *Let G/D a connected finite volume homogeneous space. A homogeneous flow $(G/D, g^{\mathbb{R}})$ is either smoothly isomorphic to a linear flow on a torus or it has countably many independent invariant distributions.*

An important feature of our argument is that in the case of *partially hyperbolic* flows we prove the stronger and more general result that any partially hyperbolic flow on any compact manifold, not necessarily homogeneous, has infinitely many distinct minimal sets (see Theorem 2.1). In particular, we have a proof of the Katok and Greenfield-Wallach conjectures in this case. We are not able to generalize this result to the finite-volume case. However, we can still prove that a partially hyperbolic *homogeneous* flow on a finite-volume manifold has countably many ergodic probability measures (see Proposition 5.3).

In the non partially hyperbolic homogeneous case, that is, in the quasi-unipotent case, by the Levi decomposition we are able to reduce the problem to flows on semi-simple and solvable manifolds. The semi-simple case is reduced to the case of $\mathrm{SL}_2(\mathbb{R})$ by an application of the Jacobson–Morozov's Lemma which states that any nilpotent element of a semi-simple Lie algebra can be embedded in an $\mathfrak{sl}_2(\mathbb{R})$-triple. The solvable case can be reduced to the nilpotent case for which our main result was already proved by the first two authors in [FF07]. In both these cases the construction of invariant distributions is based on the theory of unitary representations for the relevant Lie group (Bargmann's classification for $\mathrm{SL}(2, \mathbb{R})$ and Kirillov's theory for nilpotent Lie groups).

The paper is organized as follows. In section 2 we deal with partially hyperbolic flows on compact manifolds. In section 3 give the background on homogeneous flows that allows us to reduce the analysis to the solvable and semi-simple cases. A further reduction is to consider quasi-unipotent flows (sect. 4) and partially hyperbolic flows (sect. 5) on finite-volume non-compact manifolds; then the main theorem follows easily (sect. 6). Finally, in section 7 we state a general conjecture

on the stability of homogeneous flows and a couple of more general related open problems.

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2. Partially hyperbolic flows on compact manifolds

The goal of this section is to prove the following theorem.

Theorem 2.1. *Let M be a compact connected manifold, ϕ^t, $t \in \mathbb{R}$, a flow on M and assume ϕ^t leaves invariant a foliation $\mathcal{F}$ with smooth leaves and continuous tangent bundle, e.g. the unstable foliation of a partially hyperbolic flow. Assume also that the flow ϕ^t expands the norm of the vectors tangent to $\mathcal{F}$ uniformly. Then there are infinitely many different ϕ^t-minimal sets.*

The existence of at least one non trivial (i.e. different from the whole manifold) minimal sets goes back G. Margulis (see [Sta00], [KM96], [KSS02]) and Dani [Dan85], [Dan86]. A similar idea was already used by R. Mañé in [Mañ78] and more recently by A. Starkov [Sta00], and F. & J. Rodriguez Hertz and R. Ures [RHRHU08, Lemma A.4.2 (Keep-away Lemma)], in different contexts.

Theorem 2.1 will follow almost immediately from the next lemma.

Lemma 2.2. *Let ϕ^t be a flow like in Theorem 2.1. For any k-tuple $p_1, \dots, p_k \in M$ of points in different orbits and for any open set $W \subset M$, there exist $\epsilon > 0$ and $q \in W$ such that $d(\phi^t(q), p_i) \geq \epsilon$ for all $t \geq 0$ and for all $i = 1, \dots, k$.*

Let us show how Theorem 2.1 follows from Lemma 2.2.

Proof of Theorem 2.1. Since M is compact there is a minimal set K. Assume now by induction that there are $K_1, \dots K_k$ different minimal set, then we will show that there is a minimal set K_{k+1} disjoint from the previous ones. Let $p_i \in K_i$, $i = 1, \dots, k$ be k points and take q and $\epsilon > 0$ from Lemma 2.2. Since $d(\phi^t(q), p_i) \geq \epsilon$ for any $t \geq 0$ and for $i = 1, \dots, k$ we have that for any $i = 1, \dots, k$, $p_i \notin \omega(q)$, the omega-limit set of q. Since the K_i's are minimal, this implies that $K_i \cap \omega(q) = \emptyset$ for $i = 1, \dots, k$. Take now a minimal subset of $\omega(q)$ and call it K_{k+1}. $\square$

Let $F = T\mathcal{F}$ be the tangent bundle to the foliation with fiber at $x \in M$ given by $F(x) = T_x\mathcal{F}(x)$. We denote by d the distance on M induced by some Riemannian metric. Let X be the generator of the flow ϕ^t. Let also $E(x) = (F(x) \oplus \langle X(x) \rangle)^\perp$ be the orthogonal bundle and $\mathcal{E}_r(x) = \exp_x(B_r^E(x))$ be the image of the r ball in $E(x)$ by the exponential map. Let f be the dimension of the foliation $\mathcal{F}$ and m the dimension of M. For $r \leq r_0$ the disjoint union $\mathcal{E} := \sqcup_{x \in M} \mathcal{E}_r(x)$ is a $(m - f - 1)$-dimensional continuous disc bundle over M. Denote with $d_{\mathcal{F}}$ and $d_{\mathcal{E}}$ the distances along the leaves of $\mathcal{F}$ and $\mathcal{E}$, and let

$$\mathcal{F}_r(x) = \{y \in \mathcal{F}(x) \mid d_{\mathcal{F}}(y, x) \leq r\} \subset \mathcal{F}(x)$$

be the f-dimensional closed disc centerd at x and of radius $r > 0$. Clearly $d \leq d_{\mathcal{F}}$ and $d \leq d_{\mathcal{E}}$.

We may assume that the Riemannian metric on M is adapted so that $\mathcal{F}_r(x) \subset \phi^{-t}\mathcal{F}_r(\phi^t x)$ for all $x \in M$ and $r, t \geq 0$. In fact if g is a Riemannian metric such that $\| \phi^t_* v \|_g \geq C\lambda^t \| v \|_g$ for all $v \in F(x)$, all $x \in M$ and all $t \geq 0$, (where $\lambda > 1$), then setting $\hat{g} = \int_0^{T_0} (\phi^t)^* g \, dt$, with $T_0 = -\log_\lambda(C/2)$, we have that, for all $v \in F(x)$ and $x \in M$, the function $\| \phi^t_* v \|_{\hat{g}}$ is strictly increasing with t.

We may choose $r_1 < r_0$ such that if $r \le r_1$ then, for all $x \in M$, we have $d_{\mathcal{F}}(y,z) \le 2d(y,z)$ for any $y,z \in \mathcal{F}_r(x)$ and $d_{\mathcal{E}}(y,z) \le 2d(y,z)$ for any $y,z \in \mathcal{E}_r(x)$.

For $x \in M$, let

$$V_{\delta,r}(x) = \bigcup_{z \in \mathcal{E}_\delta(x)} \mathcal{F}_r(z).$$

There exists $r_2 \le r_1$ be such that for all $r, \delta \le r_2$ then $V_{\delta,4r}(x)$ is homeomorphic to a disc of dimension $(m-1)$ transverse to the flow.

Normalization assumption: After a constant rescaling of X we may assume that given any $x \in M$, $z, y \in \mathcal{F}(x)$ and $t \ge 1$ we have $d_{\mathcal{F}}(\phi^t(z), \phi^t(y)) \ge 4\, d_{\mathcal{F}}(z,y)$. Henceforth, in this section, we shall tacitly make this assumption.

Proof of Lemma 2.2. Let $p_1, \dots, p_k \in M$ be points belonging to different orbits and let $W \subset M$ be an open set. We shall find $r > 0$ and a point $x_0 \in W$ with $\mathcal{F}_r(x_0) \subset W$ and then construct, by induction, a sequence of points $x_n \in M$ and of iterates $\tau_n \ge 1$ satisfying, for some $\delta > 0$, the following conditions

$$(A_{n+1}) \qquad \phi^{-\tau_n}(\mathcal{F}_r(x_{n+1})) \subset \mathcal{F}_r(x_n), \qquad \text{for all } n \ge 0,$$

and

$$(B_n) \qquad \phi^{T_n} x \in \mathcal{F}_r(x_n) \implies \phi^t(x) \notin \textstyle\bigcup_i V_{\delta,2r}(p_i)\,, \qquad \text{for all } t \in [0, T_{n+1}).$$

Here we have set $T_n := \sum_{k=0}^{n-1} \tau_k$. Then defining $D_n := \phi^{-T_n}\mathcal{F}_r(x_n)$ we have $D_{n+1} \subset D_n \subset \mathcal{F}_r(x_0)$ and any point $q \in \bigcap_n D_n \subset \mathcal{F}_r(x_0) \subset W$ will satisfy the statement of the Lemma.

By the choice of an adapted metric we have

$$d_{\mathcal{F}}(\phi^t(x), \phi^t(p)) \ge d_{\mathcal{F}}(x,p)\,, \qquad \text{for all } p \in M \text{ and all } x \in \mathcal{F}(p)\,.$$

This implies that for all $i = 1, \dots, k$ any $r > 0$ and any $t \ge 0$

$$(1) \qquad x \in \mathcal{F}_{4r}(p_i) \setminus \mathcal{F}_{2r}(p_i) \implies \phi^t(x) \notin \mathcal{F}_{2r}(\phi^t(p_i)).$$

Hence there exists $\delta_0 < r_2$ such that for all $\delta < \delta_0$ and all $r \le r_2$ we have:

(i) for all $i \in \{1, \dots, k\}$,

$$\phi^{[0,1]}\big(V_{\delta,4r}(p_i) \setminus V_{\delta,2r}(p_i)\big) \cap V_{\delta,r}(p_i) = \emptyset\,.$$

The above assertion follows immediately by continuity if p_i is not periodic of minimal period less or equal to 1. If p_i is periodic of period less or equal to 1, then it follows by continuity from formula (1).

As the orbits of $p_1, \dots, p_k$ are all distinct and the set W is open, we may choose a point $x_0 \in W$ and positive real numbers $r, \delta < \delta_0$ so that the following conditions are also satisfied:

(ii) $\mathcal{F}_r(x_0) \subset W$;

(iii) for all $i, j \in \{1, \dots, k\}$, with $i \ne j$,

$$\phi^{[0,1]}\big(V_{\delta,4r}(p_i)\big) \cap \phi^{[0,1]}\big(V_{\delta,4r}(p_j)\big) = \phi^{[0,1]}\big(V_{\delta,4r}(p_i)\big) \cap \bigcup_{t \in [0,1]} \mathcal{F}_r(\phi^t x_0) = \emptyset\,.$$

If for all $t > 0$ we have $\mathcal{F}_r(\phi^t(x_0)) \cap \bigcup_i V_{\delta,r}(p_i) = \emptyset$, then $d(\phi^t(x_0), p_i) > r$ for all $i = 1, \dots k$ and all $t > 0$, proving the Lemma with $q = x_0$ and $\epsilon = r$. Thus we may assume that

$$\tau_0 := \inf\Big\{ t > 0 \;:\; \mathcal{F}_r(\phi^t(x_0)) \cap \textstyle\bigcup_i V_{\delta,r}(p_i) \ne \emptyset \Big\} < \infty$$

and define

$$\hat{x}_0 = \phi^{\tau_0}(x_0).$$

The above condition (iii) implies that $\tau_0 \geq 1$, hence by the normalisation assumption it follows that

$$\mathcal{F}_{5r}(\hat{x}_0) \subset \phi^{\tau_0}\big(\mathcal{F}_r(x_0)\big). \tag{2}$$

Assume, by induction, that points $x_k \in M$ and iterates $\tau_k \geq 1$ satisfying the conditions (A_n) and (B_n) have been constructed for all $k \in \{0, \dots, n\}$, and assume that the point $\hat{x}_n := \phi^{\tau_n}(x_n) \in M$ is such that $\mathcal{F}_r(\hat{x}_n)$ intersects non-trivially some disc $V_{\delta,r}(p_i)$. Since $V_{\delta,r}(p_i)$ is saturated by $\mathcal{F}$, it follows that $\mathcal{F}_{2r}(\hat{x}_n) \cap \mathcal{E}_\delta(p_i)$ consists of a unique point z_n with $d_{\mathcal{F}}(z_n, \hat{x}_n) \leq 2r$; we define $x_{n+1} \in \mathcal{F}(\hat{x}_n)$ as the point at distance $3r$ on the geodesic ray in $\mathcal{F}(\hat{x}_n)$ going from z_n to $\hat{x}_n$ (or any point on the geodesic ray issued from z_n if $\hat{x}_n = z_n$). Then we have

$$\mathcal{F}_r(x_{n+1}) \subset \mathcal{F}_{4r}(\hat{x}_n) \cap V_{\delta,4r}(p_i) \setminus V_{\delta,2r}(p_i)\,. \tag{3}$$

Since $\mathcal{F}_r(x_{n+1}) \subset V_{\delta,4r}(p_i)$, for all $t \in (0,1)$ we have

$$\mathcal{F}_r(\phi^t x_{n+1}) \subset \phi^t \mathcal{F}_r(x_{n+1}) \subset \phi^{[0,1]}\big(V_{\delta,4r}(p_i) \setminus V_{\delta,2r}(p_i)\big)\,.$$

By the disjointness conditions (i) and (iii), it follows that, for all $t \in (0,1]$,

$$\mathcal{F}_r(\phi^t x_{n+1}) \cap \bigcup_{i=1}^{k} V_{\delta,r}(p_i) = \emptyset\,.$$

It follows that if we define

$$\tau_{n+1} := \inf\Big\{t > 0 \;:\; \mathcal{F}_r(\phi^t(x_{n+1})) \cap \textstyle\bigcup_i V_{\delta,r}(p_i) \neq \emptyset\Big\}, \qquad \hat{x}_{n+1} = \phi^{\tau_{n+1}}(x_{n+1})$$

(assuming $\tau_{n+1} < +\infty$), then $\tau_{n+1} \geq 1$, and by the normalisation assumption and by the inclusion in formula (3) we have

$$\mathcal{F}_r(x_{n+1}) \subset \mathcal{F}_{4r}(\hat{x}_n) = \mathcal{F}_{4r}(\phi^{\tau_n} x_n) \subset \phi^{\tau_n}\big(\mathcal{F}_r(x_n)\big)$$

and by construction, having set $T_{n+2} := \sum_{k=0}^{n+1} \tau_k$, we also have

$$x \in D_{n+1} := \phi^{-T_{n+1}}(\mathcal{F}_r(x_{n+1})) \implies \phi^t(x) \notin \textstyle\bigcup_i V_{\delta,2r}(p_i)\,, \qquad \text{for all } t \in [0, T_{n+2}).$$

The inductive construction is thus completed. As we explained above we have that (D_n) is a decreasing sequence of closed sub-intervals of $\mathcal{F}_r(x_0)$ and that any point $q \in \bigcap_n D_n$ satisfies $\phi^t(q) \notin \bigcup_i V_{\delta,r}(p_i)$ for all $t \geq 0$.

The above inductive construction may fail if at some stage $n \geq 0$ we have $\tau_n = +\infty$. In this case let q be any point in $\phi^{-T_n}(\mathcal{F}_r(x_n))$. Again such a point $q \in W$ satisfies the statement of the Lemma, hence the proof is completed. □

3. Homogeneous flows

Henceforth G will be a connected Lie group and G/D a finite volume space; this means that D is a closed subgroup of G and that G/D has a finite G-invariant (smooth) measure. The group D is called the *isotropy group* of the space G/D.

With $g^{\mathbb{R}}$ we denote a one-parameter subgroup $(g^t)_{t\in\mathbb{R}}$ of G. The flow generated by this one-parameter subgroup on the finite volume space G/D will be denoted $(G/D, g^{\mathbb{R}})$ or simply $g^{\mathbb{R}}$.

Remark 3.1. Our first observation is that, in proving Theorem 1.1, we may suppose that the flow $g^{\mathbb{R}}$ is ergodic on G/D with respect to a finite G-invariant measure. This is due to the fact that the ergodic components of the flow $g^{\mathbb{R}}$ are closed subsets of G/D (see [Sta00, Thm. 2.5]). Since G/D is connected, either we have infinitely many ergodic components, in which case Theorem 1.1 follows, or the flow $g^{\mathbb{R}}$ is ergodic.

Henceforth we shall assume that the flow $g^{\mathbb{R}}$ is ergodic. Whenever convenient we may also assume that G is simply connected by pulling back the isotropy group D to the universal cover of G.

Let $G = L \ltimes R$ be the Levi decomposition of a simply connected Lie group G and let G_∞ be the smallest connected normal subgroup of G containing the Levi factor L. Let $q : G \to L$ be the projection onto the Levi factor. We shall use the following result.

Theorem 3.2 ([Dan77], [Sta87], [Sta00, Lemma 9.4, Thm. 9.5])**.** *If G is a simply connected Lie group and the flow $(G/D, g^{\mathbb{R}})$ on the finite volume space G/D is ergodic then*

- *The groups $R/R \cap D$ and $q(D)$ are closed in G and in L respectively. Thus G/D factors onto $L/\overline{q(D)} \approx G/\overline{RD}$ with fiber $R/R \cap D$. The semi-simple flow $(L/\overline{q(D)}, q(g^{\mathbb{R}}))$ is ergodic.*
- *The solvable flow $(G/\overline{G_\infty D}, g^{\mathbb{R}})$ is ergodic.*

By Theorem 3.2 it is possible to reduce the analysis of the general case to that of the semi-simple and solvable cases. In fact, the following basic result holds.

Lemma 3.3. *If the flow $(G/D, g^{\mathbb{R}})$ smoothly projects onto a flow $(G_1/D_1, g_1^{\mathbb{R}})$, via an epimorphism $p : G \to G_1$ with $D \subset p^{-1}(D_1)$, then the existence of countable many independent invariant distributions for the flow $(G_1/D_1, g_1^{\mathbb{R}})$ implies the existence of countable many independent invariant distributions for the flow $(G/D, g^{\mathbb{R}})$.*

Proof. We are going to outline two different proofs.
First Proof. Let μ denote the invariant smooth measure on G/D, let μ_1 the projected measure on G_1/D_1. For all $y \in G_1/D_1$, let μ_y denote the conditional measure on the fiber $p^{-1}\{y\} \subset G/D$ of the projection $p : G/D \to G_1/D_1$, which can be constructed as follows. Let ω be a volume form associated to μ on G/D and let ω_1 be a volume form associated to μ_1 on G_1/D_1. There exists a smooth form ν on G/D (of degree d equal to the difference in dimensions between G/D and G_1/D_1) such that $\omega = \nu \wedge p^*\omega_1$. Note that the form ν is not unique. (In fact, it is unique up to the addition of a d-form η such that $\eta \wedge p^*\omega_1 = 0$). However, the restriction of the form ν to the fiber $p^{-1}\{y\}$ is a uniquely determined volume form on $p^{-1}\{y\}$ which induces the conditional measure μ_y, for all $y \in G_1/D_1$.

Since the measures μ is G-invariant and the measure μ_1 is G_1-invariant, the family $\{\mu_y | y \in G_1/D_1\}$ is smooth and G_1-equivariant. For any smooth function $f \in C^\infty(G/D)$ let $p_\#(f) \in C^\infty(G_1/D_1)$ be the smooth function defined as

$$p_\#(f)(y) = \frac{1}{\mu_y(p^{-1}\{y\})} \int_{p^{-1}\{y\}} f(y) d\mu_y(x) , \quad \text{for all } y \in G_1/D_1 .$$

The map $p_\# : C^\infty(G/D) \to C^\infty(G_1/D_1)$ is bounded linear and surjective. In fact, it is a left inverse of the pull-back map $p^* : C^\infty(G_1/D_1) \to C^\infty(G/D)$. For any $g_1^{\mathbb{R}}$-invariant distribution $\mathcal{D} \in \mathcal{D}'(G_1/D_1)$, the following formula defines a $g^{\mathbb{R}}$-invariant distribution $p^\#\mathcal{D} \in \mathcal{D}'(G/D)$:

$$p^\#\mathcal{D}(f) = \mathcal{D}\left(p_*(f)\right) , \quad \text{for all } f \in C^\infty(G/D) .$$

The map $p^\# : \mathcal{D}'(G_1/D_1) \to \mathcal{D}'(G/D)$ is bounded linear and injective, since the map $p_\#$ is bounded linear and surjective, and by construction it maps the subspace of $g_1^{\mathbb{R}}$-invariant distributions into the subspace of $g^{\mathbb{R}}$-invariant distributions.
Second Proof. For any G_1-irreducible component H of the space $L^2(G_1/D_1)$, the pull-back $p^*(H)$ is a G- irreducible component of the space $L^2(G/D)$. Since p is an epimorphism, the space $C^\infty(p^*(H)) \subset p^*(H)$ of G-smooth vectors in $p^*(H)$ is equal to the pull-back $p^*C^\infty(H)$ of the subspace $C^\infty(H) \subset H$ of G_1-smooth vectors in H. Hence the push-forward map $p_* : C^\infty(p^*(H)) \to C^\infty(H)$ is well

defined. Assume that there exists a $g_1^{\mathbb{R}}$-invariant distribution which does not vanish on $C^\infty(H) \subset C^\infty(G_1/D_1)$, then there exists a $g_1^{\mathbb{R}}$-invariant distribution $\mathcal{D}_H$ which does not vanish on $C^\infty(H)$ but vanishes on $C^\infty(H^\perp)$ (which can be defined by projection). Let then $p^*(\mathcal{D}_H)$ the distribution defined as follows:

$$p^*(\mathcal{D}_H)(f) = \begin{cases} \mathcal{D}_H(p_* f)\,, & \text{if } f \in C^\infty(p^*(H))\,; \\ 0\,, & \text{if } f \in C^\infty(p^*(H)^\perp)\,. \end{cases}$$

By construction, the distribution $p^*(\mathcal{D}_H)$ is $g^{\mathbb{R}}$-invariant. Thus the one-parameter subgroup $g^{\mathbb{R}}$ on G/D has infinitely many independent invariant distributions whenever the one-parameter subgroup $g_1^{\mathbb{R}}$ on G_1/D_1 does. □

In dealing with solvable groups it is useful to recall the theorem by Mostow (see [Sta00, Theorem E.3])

Theorem 3.4 (Mostow). *If G is a solvable Lie group, then G/D is of finite volume if and only if G/D is compact.*

When G is semi-simple, in proving Theorem 1.1, we may suppose that G has finite center and that the isotropy group D is a lattice. This is the consequence of the following proposition.

Proposition 3.5. *Let G be a connected semi-simple group and let G/D be a finite volume space. If there exists an ergodic flow on G/D, then the connected component of the identity of D in G is normal in G. Hence we may assume that G has finite center and that D is discrete.*

Proof. We have a decomposition $G = K \cdot S$ of G as the almost-direct product of a compact semi-simple normal subgroup K and of a totally non-compact normal semi-simple group S. Let $p : G \to K_1 := G/S$ be the projection of G onto the semi-simple compact connected group K_1. Let $\bar g^t$ the flow generated by $\bar X = p_* X$ on the connected, compact, Hausdorff space $Y := K_1/\overline{p(D)}$. As Y is a homogeneous space of a compact semi-simple group, the fundamental group of Y is finite. The closure of the one-parameter group $(\exp t\bar X)_{t\in\mathbb{R}}$ in K_1 is a torus subgroup $T < K_1$; it follows that the closures of the orbits of $\bar g^t$ on Y are the compact tori $Tk\,\overline{p(D)}$, $(k \in K_1)$, homeomorphic to $T/T \cap k\,\overline{p(D)}\,k^{-1}$.

Let $g^{\mathbb{R}}$ be an ergodic flow on G/D, generated by $X \in \mathfrak{g}_0$. Since $\bar g^t$ acts ergodically on Y, the action of T on Y is transitive. In this case we have $Y = T/T \cap \overline{p(D)}$, and since the space Y is a torus with finite fundamental group, it is reduced to a point. It follows that $T < \overline{p(D)} = K_1$. Thus $p(D)$ is dense in $K_1 = G/S$ and SD is dense in G.

Let $\bar D^Z$ denote the Zariski closure of $\mathrm{Ad}(D)$ in $\mathrm{Ad}(G)$ (see Remark 1.6 in [Dan80]). By Borel Density Theorem (see [Dan80, Thm. 4.1, Cor. 4.2] and [Mor05]) the hypothesis that G/D is a finite volume space implies that $\bar D^Z$ contains all hyperbolic elements and unipotent elements in $\mathrm{Ad}(G)$. As these elements generate $\mathrm{Ad}(S)$, we have $\mathrm{Ad}(S) < \bar D^Z$, and the density of SD in G implies $\mathrm{Ad}(G) = \bar D^Z$. Since the group of $\mathrm{Ad}(g) \in \mathrm{Ad}(G)$ such that $\mathrm{Ad}(g)(\mathrm{Lie}(D)) = \mathrm{Lie}(D)$ is a Zariski-closed subgroup of $\mathrm{Ad}(G)$ containing $\bar D^Z$, we obtain that the identity component D^0 of D is a normal subgroup of G and $G/D \approx (G/D^0)/(D/D^0)$. We have thus proved that we can assume that D is discrete. We can also assume that G has finite center since D is a lattice in G and therefore it meets the center of G in a finite index subgroup of the center. This concludes the proof. □

Our proof of Theorem 1.1 considers separately the cases of quasi-unipotent and the partially hyperbolic flows. We recall the relevant definitions.

Let X be the generator of the one-parameter subgroup $g^{\mathbb{R}}$ and let $\mathfrak{g}^{\mu}$ denote the generalised eigenspaces of eigenvalue μ of $\mathrm{ad}(X)$ on $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. The Lie algebra $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}^{\mu}$ and we have $[\mathfrak{g}^{\mu}, \mathfrak{g}^{\nu}] \subset \mathfrak{g}^{\mu+\nu}$. Let

$$\mathfrak{p}^0 = \sum_{\Re\mu=0} \mathfrak{g}^{\mu}, \quad \mathfrak{p}^+ = \sum_{\Re\mu>0} \mathfrak{g}^{\mu}, \quad \mathfrak{p}^- = \sum_{\Re\mu<0} \mathfrak{g}^{\mu},$$

Definition 3.6. A flow $g^{\mathbb{R}}$ on G/D is called *quasi-unipotent* if $\mathfrak{g} = \mathfrak{p}^0$ and it is *partially hyperbolic* otherwise. Thus the flow subgroup $g^{\mathbb{R}}$ is quasi-unipotent or partially hyperbolic according to whether the spectrum of the group $\mathrm{Ad}(g^t)$ acting on $\mathfrak{g}$ is contained in $U(1)$ or not.

4. The quasi-unipotent case

We now assume that flow $g_{\mathbb{R}}$ on the finite volume space is quasi-unipotent.

4.1. **The semi-simple quasi-unipotent case.** In this subsection we assume that the group G is semi-simple (and the one-parameter subgroup $g^{\mathbb{R}}$ is quasi-unipotent).

Definition 4.1. An $\mathfrak{sl}_2(\mathbb{R})$ triple (a, n^+, n^-) in a Lie algebra $\mathfrak{g}_0$ is a triple satisfying the commutation relations

$$[a, n^{\pm}] = \pm n^{\pm}, \quad [n^+, n^-] = a.$$

We recall the Jacobson–Morozov Lemma [Jac79].

Lemma 4.2 (Jacobson–Morozov Lemma)**.** *Let n^+ be a nilpotent element in a semi-simple Lie algebra $\mathfrak{g}_0$. Assume that n^+ commutes with a semi-simple element $s \in \mathfrak{g}_0$. Then in $\mathfrak{g}_0$ we can find a semi-simple element a and a nilpotent element n^- such that (a, n^+, n^-) is an $\mathfrak{sl}_2(\mathbb{R})$ triple commuting with s.*

Given a unitary representation of (π, H) of a Lie group on a Hilbert space H, we denote by H^{∞} the subspace of C^{∞}-vectors of H endowed with the C^{∞} topology, and by $(H^{\infty})'$ its topological dual.

Lemma 4.3 ([FF03])**.** *Let U^t be a unipotent subgroup of $\mathrm{PSL}_2(\mathbb{R})$. For each non-trivial irreducible unitary representation (π, H) of $\mathrm{PSL}_2(\mathbb{R})$ there exists a distribution, i.e. an element of $D \in (H^{\infty})'$, such that $U^t D = D$.*

Proposition 4.4. *Let G be semi-simple and let D be a lattice in G. Then any quasi-unipotent ergodic subgroup $g^{\mathbb{R}}$ of G admits infinitely many $g_{\mathbb{R}}$-invariant distributions on $C^{\infty}(G/D)$.*

Proof. By Proposition 3.5 we may assume that G has finite center and that D is a lattice in G. By the Jordan decomposition we can write $g^t = c^t \times u^t$ where c^t is semi-simple and u^t is unipotent with $c^{\mathbb{R}}$, $u^{\mathbb{R}}$ commuting one-parameter subgroups of G. Since $c^{\mathbb{R}}$ is semi-simple and quasi-unipotent, its closure in G is a torus T.

By the Jacobson–Morozov lemma we find an $\mathfrak{sl}_2(\mathbb{R})$ triple (a, n^+, n^-) commuting with $c^{\mathbb{R}}$, hence with T. We let (a^t, u^t, v^t) be the corresponding one-parameter groups commuting with T and let S the subgroup generated by these flows.

It is well known that the center $Z(S)$ of S is finite and that, consequently, there exists a maximal compact subgroup of S. Indeed, the adjoint representation $\mathrm{Ad}_G|S$ of S on the Lie algebra of G, as a finite dimensional representation of S, factors through $SL(2, \mathbb{R})$, a double cover of $S/Z(S)$. The kernel of $\mathrm{Ad}_G|S$ is contained in $Z(G)$, because G is connected. Since $Z(S)$ is monogenic, we have that $Z(G)$ is a subgroup of index one or two of $Z(G)Z(S)$. It follows that $Z(S)$ is finite.

Let $K < S$ denote a maximal compact subgroup of S. Let $L^2(G/D; \mathbb{C})$ be the space of complex valued L^2 functions on G/D. The subspaces H_0 and H of $L^2(G/D; \mathbb{C})$ formed respectively by the $T_1 = T \cdot K$ and $T_2 = T \cdot Z(S)$ invariant

functions are infinite dimensional Hilbert spaces since the orbits of the subgroups T_1 and T_2 in G/D are compact and the space G/D is not a finite union of these orbits. Furthermore $H_0 \subset H$ and H is also invariant under S. We deduce that H decomposes as a direct integral or direct sum of irreducible unitary representations of S; these representations are trivial on the center $Z(S)$ and therefore define irreducible unitary representations of $\mathrm{PSL}_2(\mathbb{R})$. Since each irreducible unitary representation of $\mathrm{PSL}_2(\mathbb{R})$ contains at most one K-invariant vector and H_0 is infinite dimensional, we have that the cardinality of the set of unitary irreducible representations of $\mathrm{PSL}_2(\mathbb{R})$ appearing in H is not finite. By the previous Lemma each unitary irreducible representation H_i of $\mathrm{PSL}_2(\mathbb{R})$ contained in H contains a distribution $D_i \in (H_i^\infty)'$ which is u^t invariant; this distribution is also g^t invariant since the action of c^t on H is trivial. Since the space H^∞ coincides with the Fréchet space of C^∞ functions on G/D which are $T \cdot Z(S)$ invariant, the proposition is proved. □

4.2. **The solvable quasi-unipotent case.** In this subsection we assume that the group $G = R$ is solvable and the one-parameter subgroup $g^{\mathbb{R}}$ is quasi-unipotent and ergodic on the finite volume space R/D.

We recall the following definition.

Definition 4.5. A solvable group R is called a *class* (I) *group* if, for every $g \in R$, the spectrum of $\mathrm{Ad}(g)$ is contained in the unit circle $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$.

It will also be useful remark that if R is solvable and R/D is a finite measure space, then we may assume that R is simply connected and that D is a quasi-lattice (in the language of Auslander and Mostow, the space R/D is then a *presentation*); in fact, if $\tilde{R}$ is the universal covering group of R and $\tilde{D}$ is the pull-back of D to $\tilde{R}$, then the connected component of the identity $\tilde{D}_0$ of $\tilde{D}$ is simply connected [OV94, Them. 3.4]; hence $R/D \approx \tilde{R}/\tilde{D} \approx R'/D'$, with $R' = \tilde{R}/\tilde{D}_0$ solvable, connected and simply connected and with $D' = \tilde{D}/\tilde{D}_0$ a quasi-lattice.

We also recall the construction, originally due to Malcev and generalised by Auslander *et al.*, of the *semi-simple* or *Malcev splitting* of a simply connected connected solvable group R (see [AG66], [Aus73a], [Aus73b], [Gor80]).

A solvable Lie group G is *split* if $G = N_G \rtimes T$ where N_G is the nilradical of G and T is an Abelian group acting on G faithfully by semi-simple automorphisms.

A *semi-simple* or *Malcev splitting* of a connected simply connected solvable Lie group R is a split exact sequence

$$0 \to R \xrightarrow{m} M(R) \leftrightarrows T \to 0$$

embedding R into a split connected solvable Lie group $M(R) = N_{M(R)} \rtimes T$ such that $M(R) = N_{M(R)} \cdot m(R)$; here $N_{M(R)}$ and T are as before. The image $m(R)$ of R is normal and closed in $M(R)$ and it will be identified with R.

The semi-simple splitting of a connected simply connected solvable Lie group R is unique up to an automorphism fixing R.

Let $\mathrm{Aut}(\mathfrak{r}) \approx \mathrm{Aut}(R)$ be automorphism group of the Lie algebra $\mathfrak{r}$ of R. The adjoint representation Ad maps the group R to the solvable subgroup $\mathrm{Ad}(R) < \mathrm{Aut}(\mathfrak{r})$; since $\mathrm{Aut}(\mathfrak{r})$ is an algebraic group we may consider the Zariski closure $\mathrm{Ad}(R)^*$ of $\mathrm{Ad}(R)$. The group $\mathrm{Ad}(R)^*$ is algebraic and solvable, since it's the algebraic closure of the solvable group $\mathrm{Ad}(R)$. It follows that $\mathrm{Ad}(R)^*$ has a Levi-Chevalley decomposition $\mathrm{Ad}(R)^* = U^* \rtimes T^*$, with T^* an Abelian group of semi-simple automorphisms of $\mathfrak{r}$ and U^* the maximal subgroup of unipotent elements of $\mathrm{Ad}(R)^*$.

Let T be the image of $\mathrm{Ad}(R)$ into T^* by the natural projection $\mathrm{Ad}(R)^* \to T^*$. Since T is a group of automorphisms of R, we may form the semi-direct product

$M(R) = R \rtimes T$. By definition we have a split sequence

$$0 \to R \to M(R) \leftrightarrows T \to 0.$$

It can be proved that $M(R)$ is a split connected solvable group $N_{M(R)} \rtimes T$ and it is the semi-simple splitting of R (see . loc. cit.). We remark that the splitting $M(R) = N_{M(R)} \rtimes T$ yields a new projection map $\tau : M(R) \to T$ by writing for any $g \in M(R)$, $g = n\tau(g)$ with $n \in N_{M(R)}$ and $\tau(g) \in T$. Composing τ with the inclusion $R \to M(R)$ we obtain a surjective homomorphism $\pi : R \to T$.

It is useful to recall a part of Mostow's structure theorem for solvmanifolds, as reformulated by Auslander [Aus73a, IV.3] and [Aus73b, p. 271]:

Theorem 4.6 (Mostow, Auslander)**.** *Let D be a quasi-lattice in a simply connected, connected, solvable Lie group R, and let $M(R) = R \rtimes T$ be the semi-simple splitting of R. Then T is a closed subgroup of* $\mathrm{Aut}(\mathfrak{r})$ *and the projection $\pi(D)$ of D in T is a lattice of T.*

Lemma 4.7. *If the flow $g^{\mathbb{R}}$ is ergodic and quasi-unipotent on the finite volume solvmanifold R/D, then the group R is of class (I).*

Proof. We may assume R simply connected and connected and D a quasi-lattice. Let $M(R) = N_{M(R)} \rtimes T = R \rtimes T$ be the semi-simple splitting of R. Since the one-parameter subgroup $g^{\mathbb{R}}$ is quasi-unipotent, the closure of the projection $\pi(g^{\mathbb{R}})$ in the semi-simple factor $T < \mathrm{Aut}(\mathfrak{r})$ is a compact torus $T' < T$. The surjective homomorphism $\pi : R \to T$ induces a continuous surjection $R/D \to T/\pi(D)$. By Mostow's structure Theorem R/D is compact hence $T/\pi(D)$ is a compact torus. The orbits of T' in $T/\pi(D)$ are finitely covered by T'. But $g^{\mathbb{R}}$ acts ergodically on R/D, hence T' acts ergodically and minimally on $T/\pi(D)$. It follows that $T/\pi(D)$ consists of a single T' orbit and, since T', T are both connected and $T' < T$ and $T' \to T'/\pi(D)$ is a finite cover, we obtain that $T' = T$. Thus T consist of quasi-unipotent automorphisms of $\mathfrak{r}$, which implies that for all $g \in R$, the automorphism $\mathrm{Ad}(g)$ is quasi-unipotent. Hence the group R is of class (I). $\square$

The following theorem was first proved in [AG66, Thm. 4.4] under the hypothesis that $D \cap N_{M(R)} = D$. This amounts to suppose that D is nilpotent, which is the case when R/D supports a minimal flow, as it is proved in [Aus73b, Thm. C]. A simplification of the latter proof under the hypothesis that R/D carries an ergodic flow appears in [Sta00, Theorem 7.1].

Theorem 4.8 (Auslander, Starkov)**.** *An ergodic flow on a class (I) compact solvable manifold is smoothly isomorphic to a nilflow.*

The first two authors have proved that the main theorem holds for general nilflows, that is, that the following result holds:

Theorem 4.9 ([FF07])**.** *An ergodic nilflow which is not toral, has countably many independent invariant distributions.*

In conclusion we have

Proposition 4.10. *An ergodic quasi-unipotent flow on a finite volume solvmanifold is either smoothly conjugate to a linear toral flow or it admits countably many independent invariant distributions.*

5. Partially hyperbolic homogeneous flows

In the non-compact, finite volume case, by applying results of D. Kleinbock and G. Margulis we are able to generalize Theorem 2.1 to flows on *semi-simple* manifolds. We think that it is very likely that a general partially hyperbolic flow

on any finite volume manifold has infinitely many different minimal sets, but we were not able to prove such a general statement.

For non-compact finite volume we recall the following result by D. Kleinbock and G. Margulis [KM96] and its immediate corollary.

Theorem 5.1 (Kleinbock and Margulis)**.** *Let G be a connected semi-simple Lie group of dimension n without compact factors, Γ an irreducible lattice in G. For any partially hyperbolic homogeneous flow $g^{\mathbb{R}}$ on G/Γ, for any closed invariant set $Z \subset G/\Gamma$ of (Haar) measure zero and for any nonempty open subset W of G/Γ, we have that*

$$\dim_H(\{x \in W \mid g^{\mathbb{R}}x \textit{ is bounded and } \overline{g^{\mathbb{R}}x} \cap Z = \emptyset\}) = n.$$

Here $\dim_H$ denotes the Hausdorff dimension.

Observe that if the flow $g^{\mathbb{R}}$ is ergodic, it is enough to assume that the closed invariant set $Z \subset G/\Gamma$ be proper.

Corollary 5.2. *Under the conditions of Theorem 5.1 the flow $(G/\Gamma, g^{\mathbb{R}})$ has infinitely many different compact minimal invariant sets.*

Proposition 5.3. *Let G be a connected semi-simple Lie group and G/D a finite volume space. Assume that the flow $g^{\mathbb{R}}$ on G/D is partially hyperbolic. Then the flow $(G/\Gamma, g^{\mathbb{R}})$ has infinitely many distinct compact minimal invariant sets, and consequently infinitely many $g_{\mathbb{R}}$-invariant and mutually singular ergodic measures.*

Proof. Let $G = K \times S$ be the decomposition of G as the almost-direct product of a compact semi-simple subgroup K and of a totally non-compact semi-simple group S, with both K and S connected normal subgroups. Since the flow $(G/\Gamma, g^{\mathbb{R}})$ is partially hyperbolic S is not trivial. Since K is compact and normal, then $D' = DK = KD \subset G$ is a closed subgroup, and since $D \subset KD$, then G/KD is of finite volume. Moreover, $(G/K)/DK \sim G/KD$ is of finite volume and $G' = G/K \sim S/S \cap K$ is semi-simple without compact factor with $D' \subset G'$ a closed subgroup with G'/D' of finite volume and a projection $p : G/D \to G'/D'$. Thus we may assume that G is totally non compact, and by Proposition 3.5, that the center of G is finite and that D is a lattice.

If D is irreducible, then the statement follows immediately from Corollary 5.2. Otherwise, let G_i, for $i \in \{1, \dots, l\}$, be connected normal simple subgroups such that $G = \prod_i G_i$, $G_i \cap G_j = \{e\}$ if $i \neq j$, and let $\Gamma_i = \Gamma \cap G_i$ be an irreducible lattice in G_i, for each $i \in \{1, \dots, l\}$, with $\Gamma_0 = \prod_i \Gamma_i$ of finite index in Γ. Observe that $G/\Gamma_0 \sim \prod_i G_i/\Gamma_i$. Let $p : G/\Gamma_0 \to G/\Gamma$ be the finite-to-one covering and let $p_i : G/\Gamma_0 \to G_i/\Gamma_i$ be the projections onto the factors. Let $g_0^{\mathbb{R}}$ be the flow induced by the one-parameter group $g^{\mathbb{R}}$ on G/Γ_0 and let $g_i^{\mathbb{R}}$ be the projected flow on G_i/Γ_i, for all $i \in \{1, \dots, l\}$. Since Γ_i is an irreducible lattice in G_i, whenever $g_i^{\mathbb{R}}$ is partially hyperbolic we can apply Corollary 5.2. Since $g_0^{\mathbb{R}}$ is partially hyperbolic there is at least one $j \in \{1, \dots, l\}$ such that $g_j^{\mathbb{R}}$ is partially hyperbolic. By Corollary 5.2, the flow $g_j^{\mathbb{R}}$ has a countable family $\{K_n | n \in \mathbb{N}\}$ of distinct minimal subsets of G_j/Γ_j such that each K_n supports an invariant probability measure η_n. For all $n \in \mathbb{N}$, let us define $\hat{\mu}_n := \eta_n \times Leb$ on G/Γ_0. By construction the measures $\hat{\mu}_n$ are invariant, for all $n \in \mathbb{N}$, and have mutually disjoint supports. Finally, since the map $p : G/\Gamma_0 \to G/\Gamma$ is finite-to-one, it follows that the family of sets $\{p(K_n \times \prod_{i \neq j} G_i/\Gamma_i) | n \in \mathbb{N}\}$ consists of countably many disjoint closed sets supporting invariant measures $\mu_n := p_* \hat{\mu}_n$. The proof of the Proposition is therefore complete. ∎

6. The general case

We may now prove our main theorem.

Proof of Theorem 1.1. By Remark 3.1 we may suppose the flow ergodic. Let also assume that G is simply connected, by possibly pulling back D to the universal cover of G. Recall that by Theorem 3.2 the ergodic flow $(G/\Gamma, g^{\mathbb{R}})$ projects onto the ergodic flow $\big(L/\overline{q(D)}, q(g^{\mathbb{R}})\big)$, where L is the Levi factor of G and $q : G \to L$ the projection of G onto this factor. Assume that the finite measure space $\big(L/\overline{q(D)}\big)$ is not trivial. Then the statement of the theorem follows from Proposition 4.4 if the flow $\big(L/\overline{q(D)}, q(g^{\mathbb{R}})\big)$ is quasi-unipotent and by Proposition 5.3 if it is partially hyperbolic.

If the finite measure space $\big(L/\overline{q(D)}\big)$ is reduced to a point, then, using again Theorem 3.2, we have $G/D \approx R/R \cap D$, where R is the radical of G. We obtain in this way that our original flow is diffeomorphic to an ergodic flow on a finite volume solvmanifold. By Mostow's theorem (see Theorem 3.4), a finite volume solvmanifold is compact. Hence the statement of the theorem follows from Theorem 2.1 if the projected flow is partially hyperbolic and by Proposition 4.10 if it is quasi-unipotent. The proof is therefore complete. □

7. Open problems

We conclude the paper by stating some (mostly well-known) open problems and conjectures on the stability and the codimension of smooth flows.

Conjecture 7.1. (A. Katok) Every homogeneous flow (on a compact homogeneous space) which fails to be stable (in the sense that the range of the Lie derivative on the space of smooth functions is not closed) projects onto a Liouvillean linear flow on a torus. In this case, the flow is still stable on the orthogonal complement of the subspace of toral functions (in other words, the subspace of all functions with zero average along each fiber of the projection).

It is known that hyperbolic and partially hyperbolic, central isometric (or more generally with uniform sub-exponential central growth), accessible systems are stable (in the hyperbolic case it follows by Livshitz theory; in the partially hyperbolic accessible case see the work of A. Wilkinson [Wil08] for accessible partially hyperbolic maps and references therein). In the partially hyperbolic non-accessible case, several examples are known to be stable (see [Vee86] for toral automorphisms and [Dol05] for group extensions of Anosov). In the unipotent case, it is proved in [FF03] that $SL(2, \mathbb{R})$ unipotent flows (horocycles) on finite volume homogeneous spaces are stable and in [FF07] that the above conjecture holds for nilflows.

Problem 7.1. Classify all compact manifolds which admit uniquely ergodic flows with (a) a unique invariant distribution (equal to the unique invariant measure) up to normalization; (b) a finite dimensional space of invariant distributions.

Example of manifolds (and flows) of type (a) have been found by A. Avila, B. Fayad and A. Kocsard [AFK12]. Note that the Katok (Greenfield-Wallach) conjecture implies that in all non-toral examples of type (a) the flow cannot be stable. Recently A. Avila and A. Kocsard [AK13] have announced that they have constructed maps on the two-torus having a space of invariant distributions of arbitrary odd dimension. It is unclear whether examples of this type can be stable:

Problem 7.2. (M. Herman) Does there exists a stable flow with finitely many invariant distributions which is not smoothly conjugate to a Diophantine linear flow on a torus?

The only known example which comes close to an affirmative answer to this problem is given by generic area-preserving flows on compact higher genus surfaces [For97], [MMY05]. Such flows are generically stable and have a finite dimensional space of invariant distributions in every finite differentiability class (but not in the class of infinitely differentiable functions).

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